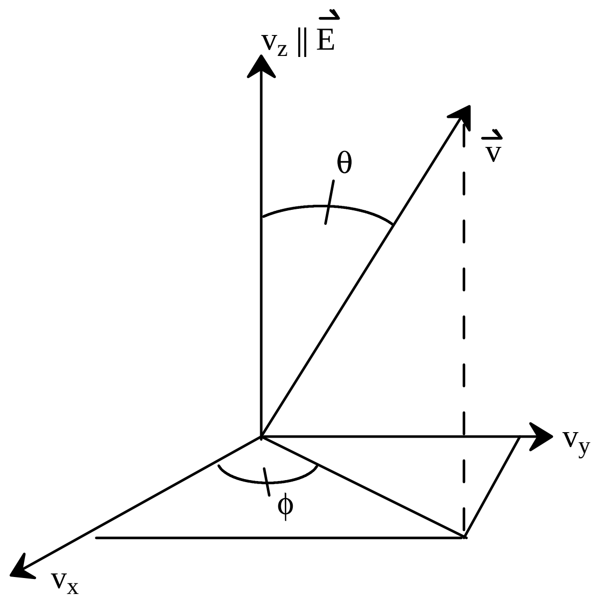
SPHERICAL HARMONIC EXPANSION

1. Expand the electron velocity distribution in a series of Legendre polynomials. Each term in the series brings in more anisotropy

$$f(\vec{v}, \vec{r}, t) = \sum_{k=0}^{\infty} P_k(\cos\theta)\, f_k(v, \vec{r}, t)$$

where $\vec{v}$ is the vector velocity and v is the speed.



a. θ is measured from the direction of $\vec{E}$, usually aligned along $\vec{v}_z$.

b. f_k is a function only of $|\vec{v}_k| = v$.

$$f_{k+n} << f_k$$

c. $P_k(\cos\theta)$ is the k^{th} Legendre polynomial

k		
0	$P_0(\cos\theta) = 1$	-Isotropic component
1	$P_1(\cos\theta) = \cos\theta$	-Drift in direction of field
2	$P_2(\cos\theta) = \frac{1}{4}(1 + 3\cos 2\theta) = \frac{1}{2}(3\cos^2\theta - 1)$	-Higher order transport

2. Keep only the first two terms

$$f(\vec{v}, \vec{r}, t) = f_0(v, \vec{r}, t) + f_1(v, \vec{r}, t)\cos\theta, \int_0^{\infty} f_0(v) 4\pi v^2 dv = 1$$

where f_0 is the isotropic component. Substitute into Boltzman's equation with $\vec{E} = E_z\hat{z}$, $\vec{a} = \frac{-q\vec{E}}{m_e}$. (See Appendix A of Cherrington.)

a. $$\frac{\partial f_0}{\partial t} + \frac{v}{3}\frac{\partial f_1}{\partial z} - \frac{qE_z}{m_e}\frac{1}{3}\frac{1}{v^2}\frac{2}{2v}\left(v^2 f_1\right) = S_0$$

b. $$\frac{\partial f_1}{\partial t} + v\frac{\partial f_0}{\partial z} - \frac{qE_z}{m_e}\frac{\partial f_0}{\partial v} = S_1$$

$$S_k = \frac{2k+1}{4\pi} \int P_k \left(\frac{\partial f}{\partial t}\right)_c d\Omega$$

3. Elastic collisions only:

For $\frac{\partial f}{\partial t} = 0, \quad \frac{\partial}{\partial z} = 0$, and elastic collisions only.

$$\text{So} = \frac{1}{2v^2} \frac{\partial}{\partial v} \left(v^2 \delta \nu_m(v) \left(\frac{kT_g}{m_e} \frac{\partial f_0}{\partial v} + v f_0 \right) \right)$$

$$\delta = \frac{2m_e}{M}, \quad \nu_m = v \cdot N \cdot \sigma_m(v)$$

Solve for S_1 from **b.** and substitute into **a.**

$$f_1 = \frac{qE_z}{m_e \nu_m} \frac{\partial f_0}{\partial v}$$

Note that f_1 is finite (e.g., anisotropic) only for $E \quad 0$.

$$\frac{-qE_z}{m_e} \frac{1}{3} \frac{1}{v^2} \frac{\partial}{\partial v} \left(v^2 \frac{qE}{m_e \nu_m} \frac{\partial f_0}{\partial v} \right) = \frac{1}{2v^2} \frac{\partial}{\partial v} \left(v^2 \delta \nu_m \left(\frac{kT_g}{m_e} \frac{\partial f_0}{\partial v} + v f_0 \right) \right)$$

Note: $\quad f_0(\varepsilon) = 4\pi \sqrt{2}\, f_0(v) \varepsilon^{1/2} / m_e^{3/2}, \quad v = \left(\frac{2\varepsilon}{m_e} \right)^{1/2}$

$$dv = \frac{d\varepsilon}{m_e v} = \frac{d\varepsilon}{(2m_e \varepsilon)^{1/2}}$$

In energy representation

$$\frac{\partial}{\partial \varepsilon} \left(\frac{q^2 E_z^2 \varepsilon}{3N\sigma_m(\varepsilon)} \left(\frac{\partial f_0}{\partial \varepsilon} \right) \right) + \frac{2m_e}{M} \frac{\partial}{\partial \varepsilon} \left(\varepsilon^2 N \sigma_m(\varepsilon) \left(f_0 + kT_g \left(\frac{\partial f_0}{\partial \varepsilon} \right) \right) \right) = 0$$

In velocity representation, solve for $\frac{\partial f_0}{\partial v}$.

$$\frac{\partial f_0}{\partial v} = \left(\frac{m_e}{kT_g} v f_0 \right) \Big/ \left(1 + \frac{2}{3} \frac{1}{\delta k T_g \nu_m^2} \cdot \frac{q^2 E^2}{m_e} \right)$$